Optimization of Flexible Wing Structures Subject to Strength and Induced Drag Constraints

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An optimization procedure for designing wing structures subject to stress, strain, and drag constraints is presented. The optimization method utilizes an extended penalty function formulation for converting the constrained problem into a series of unconstrained ones. Newton's method is used to solve the unconstrained problems. An iterative analysis procedure is used to obtain the displacements of the wing structure including the effects of load redistribution due to the flexibility of the structure. The induced drag is calculated from the lift distribution. Approximate expressions for the constraints used during major portions of the optimization process enhance the efficiency of the procedure. A typical fighter wing is used to demonstrate the procedure. Aluminum and composite material designs are obtained. The tradeoff between weight savings and drag reduction is investigated.

Introduction

THE use of mathematical programming optimization techniques for determining the minimum-mass design of structures dates back to the early 1960s.[1] Specialized algorithms such as the fully stressed design technique usually are more economical than mathematical programming when the structure is subject to a single type of constraint and usually lead to optimal or near-optimal designs. However, when a structure is to be designed subject to multiple types of constraints, mathematical programming techniques, because of their generality, are the only available tool.† Early structural design procedures based on mathematical programming techniques were limited to simple structures and a small number of design variables and thus did not have much practical application. In recent years, new and more efficient procedures have been developed[3,4] which can solve more complex design problems. Such procedures usually are based on a combination of more rapidly convergent mathematical programming algorithms and on the use of approximate analysis techniques during part of the optimization process.

One of the primary applications for mathematical programming optimization procedures is the design of aircraft wing structures that are subject to multiple types of constraints such as stress, buckling, divergence, and flutter.[3-7] The growing popularity of the use of composite materials in wing structures creates new challenges to existing optimization procedures in two major ways. First, the use of composites often results in an increased flexibility of the wing structure which leads to a redistribution of the aerodynamic loads on the wing. As the structure is resized, the aerodynamic loads are changed because the flexibility of the structure has been changed. Thus, the aerodynamic loads have to be recalculated several times during the design process, and this results in a very significant increase in the computational effort per analysis. The second challenge posed by the use of composite materials is the potential of using the increased flexibility of the wing structure to improve the performance of the structure. For example, Ref. 8 shows that the composite material distribution and ply orientations may be "tailored" to reduce the drag of the aircraft.

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†Generalized optimality criteria methods[2] also are appropriate, but they may be considered a special form of mathematical programming.

To encounter these challenges, new procedures for wing structural optimization are being developed. This paper describes such a procedure. A combination of a recently developed efficient mathematical programming technique[3] and an efficient algorithm for the combined aerodynamic-structural problem is used. One of the advantages of the combined structural-aerodynamic analysis is that aerodynamic performance constraints may be incorporated easily. This is demonstrated here by an example of the minimum weight design of a wing subject to constraints on stresses and strains in structural elements and a limit on the induced drag of the wing. These constraints are applied to a single load condition arising from a high g symmetric pull-up flight condition. The induced drag during the pull-up maneuver results in a rapid speed loss and thus limits the duration of the maneuver. The optimization procedure permits us to find the tradeoff between drag and weight so that we may decide whether it is worthwhile to reduce the drag by structural changes.

Optimization Procedure

Quadratic Extended Penalty Function

The constrained optimization problem considered herein is to find a vector of N design variables v^* that minimizes the mass $m(v)$ of a wing structure subject to constraints

$$g_i(v) \geq 0; \qquad i=1,\ldots,n \tag{1}$$

This constrained optimization problem is transformed into a series of unconstrained minimization problems by the use of a quadratic extended penalty function,[3] and the transformed problem is solved by the sequence of unconstrained minimizations technique (SUMT).[9] Each unconstrained problem is solved by using Newton's method. The resulting transformed problem is to find the minimum of a function $P(r)$ as r goes to zero, where

$$P(r) = m(v) + r\sum_{i=1}^{n} f_i(v) \tag{2}$$

and $f_i(v)$ is defined as

$$f_i = \begin{cases} 1/g_i & \text{if } g_i \geq g_0 \\ 1/g_0[(g_i/g_0)^2 - 3(g_i/g_0) + 3] & \text{if } g_i \leq g_0 \end{cases} \tag{3}$$

where g_0 is a small transition parameter, and the particular form of f_i in Eq. (3) is chosen to insure continuity up to second derivatives. As shown in Ref. 3, the transition parameter g_0 must be a function of r, and a possible relation is

$$g_0 = Cr^p; \qquad \tfrac{1}{2} \geq p > \tfrac{1}{3} \tag{4}$$

The penalty function defined by Eq. (3) is an interior penalty function $1/g_i$ extended into the infeasible domain. It permits consideration of designs outside the feasible domain, which is useful, as the approximate analysis techniques used during the optimization process often result in incursions into the infeasible domain. The continuity of the first and second derivatives of f_i is needed for the rapid convergence of the unconstrained minimization algorithm (Newton's method; see below), which requires the second derivatives of $P(r)$.

Newton's Method with Approximate Second Derivatives

To apply Newton's method with the SUMT procedure, the point v^r that minimizes the function $P(r)$ for a given value of r is found by using an iterative procedure. If v^i is an initial guess for v^r, a better approximation v^{i+1} is found from

$$v^{i+1} = v^i - \alpha H^{-1} \nabla P \tag{5}$$

where ∇P is the gradient of P, H is the matrix of second derivatives of P given by

$$H_{jk} = \frac{\partial^2 P}{\partial v_j \partial v_k} \tag{6}$$

and α is the step size from v^i to v^{i+1} found by means of a one-dimensional search in the direction of $H^{-1} \nabla P$.

Following Ref. 3, the second derivative of Eq. (2) with respect to the design variables can be written as

$$\frac{\partial^2 P}{\partial v_j \partial v_k} = \frac{\partial^2 m}{\partial v_j \partial v_k} + r \sum_{i=1}^{n} \frac{\partial^2 f_i}{\partial v_j \partial v_k} \tag{7}$$

where, from Eq. (3),

$$\begin{aligned} \frac{\partial^2 f_i}{\partial v_j \partial v_k} &= g_i^{-3} \left[2 \frac{\partial g_i}{\partial v_j} \frac{\partial g_i}{\partial v_k} - g_i \frac{\partial^2 g_i}{\partial v_j \partial v_k} \right] \quad if\ g_i \geq g_0 \\ &= g_0^{-3} \left[2 \frac{\partial g_i}{\partial v_j} \frac{\partial g_i}{\partial v_k} + g_0 \left(2 \frac{g_i}{g_0} - 3 \right) \frac{\partial^2 g_i}{\partial v_j \partial v_k} \right] \quad if\ g_i \leq g_0 \end{aligned} \tag{8}$$

Because of the factors g_i^{-3} and g_0^{-3} in Eq. (8), the main contribution to the penalty function second derivatives is from the constraints, which are nearly critical (i.e., g_i very small).‡ For these constraints, the following approximations for the second derivatives of the quadratic extended penalty function may be used:

$$\frac{\partial^2 f_i}{\partial v_j \partial v_k} = \begin{cases} 2 g_i^{-3} \dfrac{\partial g_i}{\partial v_j} \dfrac{\partial g_i}{\partial v_k} & \text{if} \quad g_i \geq g_0 \\ 2 g_0^{-3} \dfrac{\partial g_i}{\partial v_j} \dfrac{\partial g_i}{\partial v_k} & \text{if} \quad g_i \leq g_0 \end{cases} \tag{9}$$

Equation (9) includes only first derivatives of the constraint functions, so that the computational effort for obtaining the second derivatives needed for Newton's method is the same as for a first-order method. Examples given in Ref. 3 show that the number of one-dimensional searches using the foregoing procedure is independent of the number of design variables and is of the order of 20 with 4 to 5 values of r (i.e., 4 to 5 searches per r).

‡It is assumed here that g_0 is small. Note that if g_0 were not small for the first value of r, it would become small as $r \to 0$ [see Eq. (4)].

Analysis Procedure

The equilibrium equations for a wing structure subject to aerodynamic and other loads may be written in terms of a displacement vector $\boldsymbol{w}$ as

$$(K - qA)w = r \tag{10}$$

where $\boldsymbol{K}$ is a stiffness matrix, $\boldsymbol{A}$ is an aerodynamic influence coefficient matrix, and q is the dynamic pressure. The load vector $\boldsymbol{r}$ includes nonaerodynamic forces as well as aerodynamic loads due to the undeformed shape and position of the wing. Usually Eq. (10) is solved directly by elimination, for example, as in the FLEXSTAB program.[10] The direct solution of Eq. (10) by elimination is expensive because, although $\boldsymbol{K}$ is sparse and symmetric, the matrix $\boldsymbol{A}$ usually is fully populated and not symmetric. Instead, an accelerated iterative algorithm proposed here is used. The basic iterative process is well known and is defined as

$$Kw^{k+1} = r + qAw^k \tag{11}$$

This iterative process is sometimes slow for high values of the dynamic pressure q. The convergence of the iterative process is investigated by expanding the displacement vector $\boldsymbol{w}$:

$$w = \sum_{i=1}^{m} w_i u_{Ri} \tag{12}$$

where $\boldsymbol{u}_{Ri}$ are the eigenvectors of the system

$$(K - qA) w = 0 \tag{13}$$

that is, $\boldsymbol{u}_{Ri}$ satisfies the equation

$$u_{Ri} = (1/q_i) K^{-1} A u_{Ri} \tag{14}$$

and q_i is the ith eigenvalue of Eq. (13).

The error in the kth iterate w^k is expanded similarly in terms of $\boldsymbol{u}_{Ri}$:

$$e^k = w - w^k = \sum_{i=1}^{m} e_i^{\ k} u_{Ri} \tag{15}$$

Using Eqs. (10, 11, and 14), the error in the $(k+1)$th iterate is

$$e^{k+1} = w - w^{k+1} = \sum_{i=1}^{m} \left(\frac{q}{q_i} \right) e_i^{\ k} u_{Ri} \tag{16}$$

Equation (16) indicates that the ith component of the error vector e_i is being reduced in absolute magnitude by the ratio $|q/q_i|$. The error components in the directions of eigenvectors associated with the lowest q_i are reduced the most slowly. The convergence rate thus eventually is controlled by the ratio $|q/q_1|$, where q_1 is the lowest eigenvalue§ of Eq. (13). If q_1 is well separated from q_2, the next-higher eigenvalue, the convergence rate could be improved significantly if one could choose an initial approximation w^1 such that $e_1^1 = 0$ (that is, no error component in the direction of the first eigenvector). The convergence rate in this case will depend on $|q/q_2|$. An initial approximation w^1 that satisfies the condition $e_1^1 = 0$ may be obtained by using the left eigenvectors of Eq. (13), u_{Li} $i = 1,\ldots,m$, which satisfy

$$u_{Li}^H (K - q_i A) = 0 \tag{17}$$

where H denotes the Hermitian transpose.

It is easy to check that the two sets of eigenvectors are biorthogonal with respect to $\boldsymbol{K}$, that is,

§For high-aspect-ratio wings, q_1 usually is real and is called the divergence dynamic pressure; however, for some low-aspect-ratio wings, q_1 is complex.

$$u_{Li}^H K u_{Rj} = 0 \text{ if } q_i \neq q_j \tag{18}$$

using Eqs. (10, 12, 15, 17, and 18), it can be shown that we achieve $e_1^I = 0$ by taking

$$w^I = \frac{q_I}{q_I - q} \frac{u_{LI}^H r}{u_{LI}^H K u_{RI}} u_{RI} \tag{19}$$

which involves only the calculation of the first right eigenvector and first left eigenvector of Eq. (13). A similar result is obtained for the case when the lowest eigenvalue q_I is complex so that q_2 is the complex conjugate of q_I. In that case, both e_I^I and e_2^I must be zero to obtain an improvement in the convergence rate. It may be shown that one has to take for w^I twice the real value of w^I obtained from Eq. (19).

The preceding method for accelerating the convergence of the iterative process is particularly useful when Eq. (10) has to be solved for several right-hand-side vectors. This is the case in the present design procedure because the derivatives of the displacements with respect to the structural design variables satisfy Eq. (10) with different right-hand-side vectors (see next section). Additionally, when q_I is real, its value is of interest by itself because it is the divergence dynamic pressure, and the design has to satisfy the constraint $q < q_I$.

Constraint Evaluation and Approximation

Constraint Evaluation

The constraints considered in this work arise from one flight condition: a pull-up maneuver. The equations of equilibrium of the wing structure are given in the Appendix and are reduced to the standard form of Eq. (10). The Appendix also contains the equations defining the derivatives of the displacements with respect to the design variables. These have the same form as Eq. (10), that is,

$$(K - qA) \frac{\partial w}{\partial v_i} = r_i \tag{20}$$

Equation (20) is solved by the same iterative process [Eq. (11)] used to solve for the displacements. Stress, strain, minimum gage, and induced drag constraints are imposed on the design. The stresses and strains are obtained readily from the displacements so that we need discuss only the induced drag.

The induced drag on a wing depends only on the distribution of the lift. The lift force distribution is obtained readily from the displacements by using the aerodynamic influence coefficients. Because the induced drag depends only on the spanwise distribution of the lift, it is possible to integrate the lift across the chord and use a lifting line theory to calculate the induced drag (e.g., Ref. 11, Chap. 11). At any spanwise station y along the wing, the lift force per unit span $\ell(y)$ is related to the circulation $\Gamma(y)$ by

$$\ell(y) = \rho V \Gamma(y) \tag{21}$$

where ρ is the air density and V the freestream speed. The circulation can be expanded in a Fourier series

$$\Gamma(y) = 2bV \sum_{I}^{\infty} A_n \sin n\theta \tag{22}$$

where b is the wing semispan, and θ is defined as

$$\theta = \arccos(y/b) \qquad 0 \leq \theta \leq \pi \tag{23}$$

and the coefficient A_n is obtained from

$$A_n = \left(\frac{2}{\pi}\right) \int_0^\pi \left[\frac{\Gamma(\theta)}{2bV}\right] d\theta = \left(\frac{I}{\pi b \rho V^2}\right) \int_0^\pi \ell(\theta)\, d\theta \tag{24}$$

The induced drag D then is obtained[11] from the coefficients A_n:

$$D = \tfrac{1}{2}\pi\rho b^2 V^2 \sum_{I}^{\infty} n A_n^2 \tag{25}$$

The Fourier coefficients are obtained by using cubic spline interpolation for the lift force and a 24-point Gaussian integration formula. Because of the symmetry in the spanwise direction of the lift distribution, only odd coefficients are nonzero. Derivatives of the induced drag are obtained by differentiating Eq. (25) with respect to a design variable v_i:

$$\frac{\partial D}{\partial v_i} = \pi\rho b^2 V^2 \sum_{I}^{\infty} n A_n \frac{\partial A_n}{\partial v_i} \tag{26}$$

where

$$\frac{\partial A_n}{\partial v_i} = \left(\frac{I}{\pi b \rho V^2}\right) \int_0^\pi \frac{\partial \ell(\theta)}{\partial v_i} d\theta \tag{27}$$

and the derivatives of the lift are obtained from the derivatives of the displacements in the same way that the lift is obtained from the displacements.

Constraint Selection and Approximation

An exact analysis including an evaluation of all constraints is performed once for each one-dimensional search. On the basis of this analysis, a list of critical and near-critical constraints is selected, and only these constraints are included in the penalty function sum [Eq. (2)]. Furthermore, only the derivatives of these critical and near-critical constraints are evaluated and used to calculate a search direction for the one-dimensional search. Additionally, these derivatives are used to obtain approximate expressions for evaluating the constraints along the one-dimensional searches. Two types of

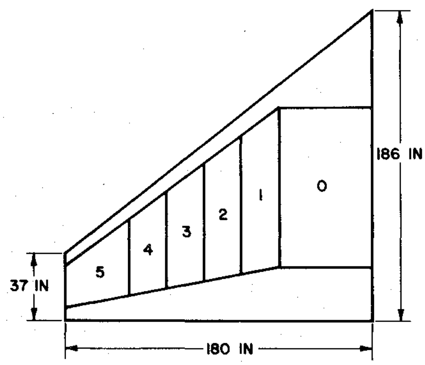

Fig. 1 Aerodynamic planform and segment definition.

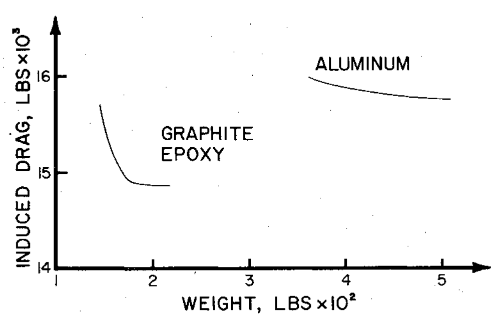

Fig. 2 Induced drag vs weight for composite and aluminum designs.

Table 1 Definition of aerodynamic coefficients and of pull-up maneuver[a]

Aircraft weight	W	21,000 lb
Cruise dynamic pressure	q_c	6. psi
Maneuver load factor	n_A	7.33
Pitch angular acceleration	$\ddot{\theta}$	0
Pitch angular speed	$\dot{\theta}$	0
Angle-of-attack rate	$\dot{\alpha}$	0
Reference area	S	40,300 in.^{2}
Reference chord	c	186 in.
Aircraft lift coefficient at $\alpha=0$	C^*_{L0}	0.0023
Slope of aircraft lift coefficient	$C_{L\alpha}$	4.4
Lift coefficient per unit elevator deflection	$C_{L\delta}$	0.5
Aircraft moment coefficient at $\alpha=0$	C_{M0}	0.0053
Slope of aircraft moment coefficient	$C_{M\alpha}$	0.5
Moment coefficient per unit elevator deflection	$C_{M\delta}$	-0.62

[a] Aerodynamic coefficients in cruise and in the pull-up maneuver were taken to be the same.

approximate expressions for the constraint functions $g(v)$ are used. For the drag constraint, a linear approximation is employed, that is, if v_0 denotes the design point where the one-dimensional search starts, then

$$g(v)=g(v_0)+\sum_{1}^{N}\frac{\partial g(v_0)}{\partial v_k}(v_k-v_{0k}) \tag{28}$$

where v_k denotes the kth design variable, $k=1,\dots,N$. For the stress and displacement constraints, the approximation is based on the assumption (which is exactly true for determinate structures) that the constraint functions are a linear function of the inverse of the thickness of the structural elements.[12] The design variable v_k in the optimization procedure is the variable thickness above minimum gage, so that the total thickness t_k is given by

$$t_k=t_{\min_k}+v_k \tag{29}$$

where $t_{\min_k}$ is the minimum gage for this design variable.

Using the assumption that $g(v)$ is linear in $1/t_k$, we obtain

$$g(v)=g(v_0)+\sum_{1}^{N}t_k(v_0)\frac{\partial g(v_0)}{\partial v_k}\left(1-\frac{t_k(v_0)}{t_k(v)}\right) \tag{30}$$

Results and Discussion

A typical fighter wing (Fig. 1) was used to demonstrate the optimization procedure. The airfoil is biconvex symmetric airfoil with $t/c=0.04$. The computer program used for performing the calculations for the examples given in this paper is a modification of the WIDOWAC program.[13] In the program, a wing structure is modeled by a finite-element representation that includes rod elements, constant-strain membrane elements for cover panels, and shear web elements for ribs and spars. The airfoil is assumed to be symmetric, so that only the upper half of the wing structure was modeled. For the purpose of optimization, the primary structure was divided into six constant thickness segments, as shown in Fig. 1. The zero segment represents the fuselage and carry-through structure and was not changed during the optimization process. Two designs were considered: an all-aluminum design, and a design using graphite/epoxy composite material for the cover panels and aluminum for the ribs and spars in both designs. For the all-aluminum design, 31 membrane elements were used to model the cover panels. For the composite design, each of the cover elements was replaced by three elements representing 0 deg, 90 deg, and combination ±45 deg ply orientations, for a total of 93 membrane elements. The 0 deg fibers were oriented in the spanwise direction. The Von Mises stress criterion with a yield stress of 40,000 psi was used for the aluminum parts, and maximum strain constraints ($\epsilon_1<0.0095$, $\epsilon_2<0.0045$, $\epsilon_{12}<0.009$)¶ were imposed on each composite lamina. A single load condition resulting from a 7.33 g pull-up maneuver at a weight of 21,000 lb and dynamic pressure $q=8$ psi was considered. The aerodynamic coefficients used in the analysis are given in Table 1.

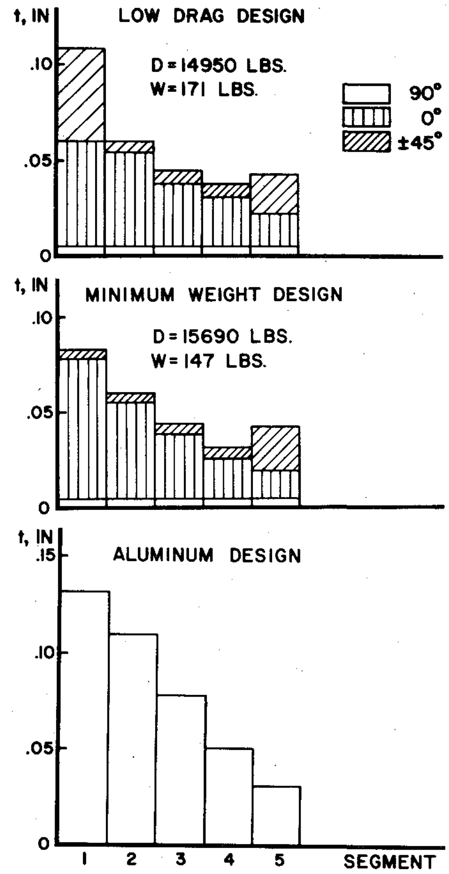

Fig. 3 Thickness distributions for aluminum and composite designs.

For the aluminum design, the thicknesses of the cover panels for segments 1 to 5 were used as design variables. Additionally, five design variables were assigned to the thickness of five groups of shear web elements. For the composite design, the thickness for each ply orientation in a segment was used as a design variable, with the shear webs represented as before by five design variables. Altogether, the all-aluminum design was optimized using 10 design variables, and the composite design was optimized using 20 design variables. Minimum gages were 0.02 in. for aluminum panel and 0.005 in. for each ply orientation.

The two designs were optimized first without applying any drag constraints. The minimum weight for the all-aluminum design was 362.3 lb** (two wings), of which 113.1 lb were due to the shear elements; induced drag was 15,990 lb. The minimum weight for the composite design was 147.1 lb, of which 58.6 lb were due to the shear elements, and the induced drag was 15,690 lb. The two designs then were optimized applying progressively more severe drag constraints. The dependence of the minimum weight on the drag constraint is shown in Fig. 2.

¶In the final designs, each laminate included all four ply orientations, so that the maximum strain in any direction was limited effectively to 0.0045.

**Only the primary structure was included.

For both aluminum and composite material, the curve has a well-defined point of highest drag which is obtained by optimizing the structure without applying any drag constraints. There is also a lowest drag, which cannot be reduced at whatever weight penalty. It is very difficult to find this lowest drag precisely because when the drag constraint is close to the lowest possible drag the feasible domain is very small and the optimization procedure converges very slowly.†† For the composite design, the drag may be reduced to 14,950 lb without excessive weight penalty (weight increases by about 24 lb), but any further reduction of the drag is very costly in terms of added weight. For the aluminum design, there was much less variation of the induced drag. The lowest induced drag obtained was 15,760 lb with a weight of 509 lb; thus, for the aluminum design and induced drag, variation was a maximum of 230 lb or about 1.5% of the total induced drag, whereas for the more flexible composite design it was 820 lb or 5.2%. It is possible that substantially higher variations in the induced drag may be obtained by using more design variables. Also, other means (such as use of control surfaces) may be available for changing the lift distribution and the resultant drag during maneuver. A curve such as the one in Fig. 2 may be used to decide which technique provides the smaller weight penalty.

The thickness distribution (including minimum gages) of the cover panels for the minimum weight designs and a low drag composite design are given in Fig. 3. As may be expected, the spanwise (0 deg) ply orientation is responsible for most of the composite design weight. The change in thickness from root to tip is much more pronounced in the aluminum design than in the composite design. This might be due to the higher tip loading caused by local deformation in the more flexible composite design. (The tip displacement was 25% larger in the composite than in the aluminum design.) The number of analyses required for the optimization was of the order of 15 to 25, and the computer time was typically 300 to 400 CPU sec on the UNIVAC 1108 computer (except for drag constraints very close to the lowest possible drag).

Concluding Remarks

An optimization procedure for designing flexible wing structures subject to stress, strain, and drag constraints was presented. The method was demonstrated on a typical fighter wing subject to stress, strain, and drag constraints. It was shown that the procedure may be used to establish the tradeoff between weight and drag. It also was shown that composite materials designs offer greater opportunities of reducing the drag of the aircraft by structural changes than all metal designs. It should be stressed that the results obtained for the fighter wing serve mainly as a demonstration of the possibilities of aerodynamic performance constraints. For actual applications, several load cases should be considered, as well as other constraints, such as flutter or panel buckling.

Appendix: Aeroelastic Equilibrium of a Wing Structure

A. Basic Equations

Using a finite-element representation of a wing structure, the displacement vector $\boldsymbol{w}$ of the wing middle surface from its undeformed shape is governed by the equation

$$\boldsymbol{K}\,\boldsymbol{w}=\boldsymbol{f} \tag{A1}$$

where $\boldsymbol{K}$ is the stiffness matrix of order m. The vector of external forces $\boldsymbol{f}$ includes the inertial force $\boldsymbol{i}$, aerodynamic force $\boldsymbol{a}$, and other (e.g., engine) forces $\boldsymbol{f}_0$:

$$\boldsymbol{f}=\boldsymbol{f}_0+\boldsymbol{i}+\boldsymbol{a} \tag{A2}$$

We limit ourselves to vertical motion of the aircraft, so that the inertial forces depend on the load factor n and pitching acceleration $\ddot{\theta}$:

$$\boldsymbol{i}=n\boldsymbol{i}_n+\ddot{\theta}\boldsymbol{i}_\theta \tag{A3}$$

where

$$n=L/W \tag{A4}$$

$$\ddot{\theta}=M/I \tag{A5}$$

and L,M,W,I are the total aircraft lift, pitching moment, weight, and moment of inertia, respectively.

The aerodynamic force on the wing is a function of the shape of the wing middle surface and the angle of attack α, its derivative $\dot{\alpha}$, and the pitch angular velocity $\dot{\theta}$:

$$\boldsymbol{a}=q[\boldsymbol{A}(\boldsymbol{u}_g+\boldsymbol{w})+\alpha\boldsymbol{a}_\alpha+\dot{\alpha}\boldsymbol{a}_{\dot{\alpha}}+\dot{\theta}\boldsymbol{a}_{\dot{\theta}}] \tag{A6}$$

where q is the dynamic pressure, $\boldsymbol{A}$ is an aerodynamic influence coefficient matrix, $\boldsymbol{u}_g$ is the shape of the undeformed wing middle surface, and $\boldsymbol{a}_\alpha$, $\boldsymbol{a}_{\dot{\alpha}}$, and $\boldsymbol{a}_{\dot{\theta}}$ are the aerodynamic force vectors per unit α, $\dot{\alpha}$, and $\dot{\theta}$, respectively. The vector $\boldsymbol{a}$ results in a total wing lift force L_W and pitching moment M_W. Symbolically, the operation of obtaining the total lift of a force vector acting on the wing or the total moment of such a force vector may be written as a scalar product introducing the vectors $\boldsymbol{\ell}$ or $\boldsymbol{m}$, that is,

$$L_W=\boldsymbol{\ell}^T\boldsymbol{a}=q[\boldsymbol{\ell}^T\boldsymbol{A}(\boldsymbol{u}_g+\boldsymbol{w})+\boldsymbol{\ell}^T(\alpha\boldsymbol{a}_\alpha+\dot{\alpha}\boldsymbol{a}_{\dot{\alpha}}+\dot{\theta}\boldsymbol{a}_{\dot{\theta}})] \tag{A7}$$

$$M_W=\boldsymbol{m}^T\boldsymbol{a}=q[\boldsymbol{m}^T\boldsymbol{A}(\boldsymbol{u}_g+\boldsymbol{w})+\boldsymbol{m}^T(\alpha\boldsymbol{a}_\alpha+\dot{\alpha}\boldsymbol{a}_{\dot{\alpha}}+\dot{\theta}\boldsymbol{a}_{\dot{\theta}})] \tag{A8}$$

The total aircraft lift and pitching moment may be written as

$$L=qS(C_{L0}+C_{LW}+\alpha C_{L\alpha}+\delta C_{L\delta}+\dot{\alpha}C_{L\dot{\alpha}}+\dot{\theta}C_{L\dot{\theta}}) \tag{A9}$$

$$M=qSc(C_{M0}+C_{MW}+\alpha C_{M\alpha}+\delta C_{M\delta}+\dot{\alpha}C_{M\dot{\alpha}}+\dot{\theta}C_{M\dot{\theta}}) \tag{A10}$$

where S is a reference area, c is a reference length, and C_{L0}, $C_{L\alpha}$, $C_{L\delta}$, $C_{L\dot{\alpha}}$, $C_{L\dot{\theta}}$, C_{M0}, $C_{M\alpha}$, $C_{M\delta}$, $C_{M\dot{\alpha}}$, $C_{M\dot{\theta}}$ are lift and moment coefficients of the aircraft for $\boldsymbol{u}_g+\boldsymbol{w}=0$, and

$$C_{LW}=[\boldsymbol{\ell}^T\boldsymbol{A}(\boldsymbol{u}_g+\boldsymbol{w})]/S \tag{A11}$$

$$C_{MW}=[\boldsymbol{m}^T\boldsymbol{A}(\boldsymbol{u}_g+\boldsymbol{w})]/Sc \tag{A12}$$

B. Cruise Conditions

The jig shape of the wing middle surface may be determined from the requirement that it will deform into a prespecified shape $\boldsymbol{u}_C$ during cruise conditions, a shape that usually is dictated by aerodynamic considerations. At cruise, we have $n=1$, $\dot{\theta}=\ddot{\theta}=0$, $\boldsymbol{u}_g+\boldsymbol{w}=\boldsymbol{u}_C$, $\dot{\alpha}=0$.

From Eqs. (A11) and (A12) we have

$$C_{LW,C}=\boldsymbol{\ell}^T\boldsymbol{A}\;\boldsymbol{u}_C/S \tag{A13}$$

$$C_{MW,C}=\boldsymbol{m}^T\boldsymbol{A}\;\boldsymbol{u}_C/Sc \tag{A14}$$

The angles of attack at cruise α_C and elevator displacement δ_C are obtained from Eqs. (A9) and (A10). Since at cruise

††This problem could be avoided by including the drag as part of the objective function rather than as a constraint. This would require, however, the calculation of second derivatives of the drag [see Eq. (7)].

$L=W, M=0,$

$$\alpha_C = \frac{[(W/q_C S) - C_{L0,C} - C_{LW,C}]C_{M\delta,C} + (C_{M0,C} + C_{MW,C})C_{L\delta,C}}{C_{L\alpha,C}C_{M\delta,C} - C_{M\alpha,C}C_{L\delta,C}} \tag{A15}$$

$$\delta_C = -\frac{[(W/q_C S) - C_{L0,C} - C_{LW,C}]C_{M\alpha,C} + (C_{M0,C} + C_{MW,C})C_{L\alpha,C}}{C_{L\alpha,C}C_{M\delta,C} - C_{M\alpha,C}C_{L\delta,C}} \tag{A16}$$

The jig shape u_g now is found from Eqs. (A1-A3 and A6):

$$u_g = u_C - K^{-1}[i_n + f_{0,C} + q_C(Au_C + \alpha_C a_\alpha)] \tag{A17}$$

C. Pull-Up Maneuver

With u_g calculated by Eq. (A17), we may use Eqs. (A1-A12) to calculate the displacement w for any flight condition. This is examplified now for a pull-up maneuver ($n=n_A$, $\ddot{\theta}=0$, $\dot{\theta}=\dot{\theta}_A$, $\dot{\alpha}=\dot{\alpha}_A$). In the following, it is understood that all of the aerodynamic coefficients, the dynamic pressure, and f_0 take the appropriate values for that maneuver. From Eqs. (A1-A12), we may obtain the following set of $m+2$ equations for w, α, and δ:

$$\begin{bmatrix} K-qA & qb_{01} & 0 \\ qb_{10}^T & qb_{11} & qb_{12} \\ qb_{20}^T & qb_{21} & qb_{22} \end{bmatrix} \begin{Bmatrix} w \\ \alpha \\ \delta \end{Bmatrix} = \begin{Bmatrix} r_0 \\ r_1 \\ r_2 \end{Bmatrix} \tag{A18}$$

where

$$b_{01} = -a_\alpha, \quad b_{10} = A^T \ell, \quad b_{11} = SC_{L\alpha}, \quad b_{12} = SC_{L\delta} \tag{A19a}$$

$$b_{20} = A^T m, \quad b_{21} = ScC_{M\alpha}, \quad b_{22} = ScC_{M\delta} \tag{A19b}$$

$$r_0 = f_0 + n_A i_n + q(Au_g + a_{\dot{\alpha}}\dot{\alpha}_A + a_{\dot{\theta}}\dot{\theta}_A) \tag{A19c}$$

$$r_1 = n_A W - q\ell^T A u_g - qS(C_{L0} + C_{L\dot{\alpha}}\dot{\alpha}_A + C_{L\dot{\theta}}\dot{\theta}_A) \tag{A19d}$$

$$r_2 = -qm^T A u_g - qSc(C_{M0} + C_{M\dot{\alpha}}\dot{\alpha}_A + C_{M\dot{\theta}}\dot{\theta}_A) \tag{A19e}$$

The last two equations may be eliminated to yield

$$(K - q\hat{A})w = \hat{r} \tag{A20}$$

$$\hat{A} = A + (1/\Delta)(b_{22}b_{01}b_{10}^T - b_{12}b_{01}b_{20}^T)$$

$$\hat{r} = r_0 - (1/\Delta)(b_{22}r_1 - b_{12}r_2)b_{01}$$

$$\Delta = b_{11}b_{22} - b_{12}b_{21} \tag{A21}$$

The stresses are obtained from the displacements by using the stress matrix S, that is,

$$\sigma = Sw \tag{A22}$$

D. Derivatives with Respect to Design Variables

The derivatives of the displacements with respect to a design variable v may be obtained by differentiating Eq. (A20)

$$\left(\frac{\partial K}{\partial v} - q\frac{\partial \hat{A}}{\partial v}\right)w + (K - q\hat{A})\frac{\partial w}{\partial v} = \frac{\partial \hat{r}}{\partial v} \tag{A23}$$

or

$$(K - q\hat{A})\frac{\partial w}{\partial v} = \Delta r \tag{A24}$$

where

$$\Delta r = \frac{\partial \hat{r}}{\partial v} - \left(\frac{\partial K}{\partial v} - q\frac{\partial \hat{A}}{\partial v}\right)w \tag{A25}$$

If v is a structural design variable which does not affect the aerodynamic coefficients then

$$\frac{\partial \hat{A}}{\partial v} = 0 \tag{A26}$$

and, from Eqs. (A19) and (A21)

$$\frac{\partial \hat{r}}{\partial v} = q\left[A\frac{\partial u_g}{\partial v} + \frac{1}{\Delta}(b_{22}\ell^T - b_{12}m^T)A\frac{\partial u_g}{\partial v}b_{01}\right] + n_A\frac{\partial i_n}{\partial v} = q\hat{A}\frac{\partial u_g}{\partial v} + n_A\frac{\partial i_n}{\partial v} \tag{A27}$$

The derivative $\partial u_g/\partial v$ is obtained by differentiating Eq. (A17)

$$\frac{\partial u_g}{\partial v} = K^{-1}\left[\frac{\partial K}{\partial v}(u_C - u_g) - \frac{\partial i_n}{\partial v}\right] \tag{A28}$$

Equation (A24) for $\partial w/\partial v$ is of the same form as Eq. (A20) and may be solved, therefore, by the same technique.

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